1) $Z(p)$ must be a positive real function of p;
2) $m_1(p)m_2(p)-n_1(p)n_2(p)=C(p^2-1)^n$.

Condition 2 implies that both numerator and denominator are of degree n and it is readily argued that an impedance function formed by terminating a section of transmission line in an indeterminant impedance function will remain indeterminant. Furthermore if $Z(p)$ is normalized so that the coefficient of p^n in its denominator is unity then C equals the terminating resistance.

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Vector Formulations for the Field Equations in Anisotropic Waveguides*

In the following we will exhibit vector formulations for the equations determining the different components of the electromagnetic field in a source-free uniform waveguide. All results will be stated without proof. The derivations are given elsewhere.[1] The vector formulations given below are applicable to uniform waveguides containing anisotropic media restricted only by the requirement that the permittivity ($\boldsymbol{\varepsilon}$) and permeability ($\boldsymbol{\mu}$) dyadics be independent of the axial coordinate z. For uniform waveguides (with the indicated restriction on $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$) we consider solutions to the Maxwell equations which display characteristic time and z dependence of the form $\exp i(\kappa z-\omega t)$. This assumption permits us to eliminate the z and t dependence from the Maxwell equations and rewrite these as:

$$\begin{bmatrix} \omega\boldsymbol{\epsilon} & -\nabla_t\times\mathbf{1}-i\kappa\mathbf{z}_0\times\mathbf{1}_t \\ -\nabla_t\times\mathbf{1}-i\kappa\mathbf{z}_0\times\mathbf{1}_t & \omega\boldsymbol{\mu}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{E}\\ i\mathbf{H}\end{bmatrix}=0. \quad (1)$$

Here, as in all the matrix equations which follow, dot product multiplication is to be understood for the products of dyadics and vectors. In (1), $\mathbf{E}$ and $\mathbf{H}$ are, respectively, the steady-state electric and magnetic fields; ∇_t is the transverse gradient operator; $\mathbf{z}_0$ is the unit vector in the axial direction; $\mathbf{1}$ is the unit dyadic; and $\mathbf{1}_t$ is the unit transverse dyadic:

$$\mathbf{1}_t=\mathbf{1}-\mathbf{1}_z=\mathbf{1}-\mathbf{z}_0\mathbf{z}_0. \quad (2)$$

It is well known that the transverse field components, $\mathbf{E}_t$ and $\mathbf{H}_t$, constitute the independent field components. To eliminate the dependent longitudinal components from (1) it is convenient to express, *e.g.*, the $\boldsymbol{\varepsilon}$ dyadic as

$$\boldsymbol{\varepsilon}\rightarrow\begin{bmatrix}\boldsymbol{\varepsilon}_t & \boldsymbol{\varepsilon}_{tz}\\ \boldsymbol{\varepsilon}_{zt} & \epsilon_z\end{bmatrix} \quad (3)$$

where $\boldsymbol{\varepsilon}_t$ is a transverse dyadic, $\boldsymbol{\varepsilon}_{tz}$ and $\boldsymbol{\varepsilon}_{zt}$ are vectors, and ϵ_z is a scalar; *i.e.*,

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_t+\epsilon_z\mathbf{1}_z+\mathbf{z}_0\boldsymbol{\varepsilon}_{zt}+\boldsymbol{\varepsilon}_{tz}\mathbf{z}_0. \quad (4)$$

A similar representation is chosen for the $\boldsymbol{\mu}$ dyadic. It can then be shown that the (independent) transverse field components satisfy the following pair of (coupled) second-order differential equations (transverse vector eigenvalue problem):

$$\begin{bmatrix}\left(\omega\boldsymbol{\varepsilon}_t-\frac{1}{\omega}\nabla_t\times\mathbf{z}_0\frac{1}{\mu_z}\mathbf{z}_0\times\nabla_t-\frac{\omega}{\epsilon_x}\boldsymbol{\varepsilon}_{tz}\boldsymbol{\varepsilon}_{zt}\right) & \left(\frac{\boldsymbol{\varepsilon}_{tz}}{\epsilon_z}\mathbf{z}_0\times\nabla_t+\nabla_t\times\mathbf{z}_0\frac{\boldsymbol{\mu}_{zt}}{\mu_z}-i\kappa\mathbf{z}_0\times\mathbf{1}_t\right)\\ \left(\frac{\boldsymbol{\mu}_{tz}}{\mu_z}\mathbf{z}_0\times\nabla_t+\nabla_t\times\mathbf{z}_0\frac{\boldsymbol{\varepsilon}_{zt}}{\epsilon_z}-i\kappa\mathbf{z}_0\times\mathbf{1}_t\right) & \left(\omega\boldsymbol{\mu}_t-\frac{1}{\omega}\nabla_t\times\mathbf{z}_0\frac{1}{\epsilon_z}\mathbf{z}_0\times\nabla_t-\frac{\omega}{\mu_z}\boldsymbol{\mu}_{tz}\boldsymbol{\mu}_{zt}\right)\end{bmatrix}\begin{bmatrix}\mathbf{E}_t\\ i\mathbf{H}_t\end{bmatrix}=0. \quad (5)$$

Once solutions to (5) are obtained, the corresponding longitudinal field components can be determined from a knowledge of the transverse components via

$$\begin{bmatrix}E_z\\ iH_z\end{bmatrix}=\begin{bmatrix}-\frac{1}{\epsilon_z}\boldsymbol{\varepsilon}_{zt} & \frac{1}{\omega\epsilon_z}\mathbf{z}_0\times\nabla_t\\ \frac{1}{\omega\mu_z}\mathbf{z}_0\times\nabla_t & -\frac{1}{\mu_z}\boldsymbol{\mu}_{zt}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{E}_t\\ i\mathbf{H}_t\end{bmatrix} \quad (6)$$

In general, to obtain solutions to the transverse vector eigenvalue problem (5) is a formidable task. We recall that even in the case of isotropic waveguides such solutions are usually obtained by replacing the vector eigenvalue problem by a pair of scalar eigenvalue problems whose eigenfunctions are (except in the case of TEM modes) proportional to the longitudinal field components. A similar technique may be employed in the general anisotropic situation under consideration here. It can be shown that the transverse field components are derivable from the longitudinal field components via

$$D(\kappa)\begin{bmatrix}\mathbf{E}_t\\ i\mathbf{H}_t\end{bmatrix}=\mathfrak{A}\mathfrak{B}\begin{bmatrix}E_z\\ iH_z\end{bmatrix} \quad (7)$$

where

$$D(\kappa)=\kappa^4+\omega^2\kappa^2\,\mathrm{Tr}\,(\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t)+\omega^4\Delta_\epsilon\Delta_\mu, \quad (8)$$

$$\mathfrak{A}=k^2\Delta_\epsilon\Delta_\mu\begin{bmatrix}\omega\boldsymbol{\varepsilon}_t^{-1} & i\kappa\boldsymbol{\varepsilon}_t^{-1}\cdot\mathbf{z}_0\times\boldsymbol{\mu}_t^{-1}\\ i\kappa\boldsymbol{\mu}_t^{-1}\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t^{-1} & \omega\boldsymbol{\mu}_t^{-1}\end{bmatrix}+\kappa^2\begin{bmatrix}\omega\mathbf{z}_0\times\boldsymbol{\mu}_t\times\mathbf{z}_0 & -i\kappa\mathbf{z}_0\times\mathbf{1}_t\\ -i\kappa\mathbf{z}_0\times\mathbf{1}_t & \omega\mathbf{z}_0\times\boldsymbol{\varepsilon}_t\times\mathbf{z}_0\end{bmatrix}, \quad (9)$$

$$\mathfrak{B}=\begin{bmatrix}-\omega\boldsymbol{\varepsilon}_{tz} & \nabla_t\times\mathbf{z}_0\\ \nabla_t\times\mathbf{z}_0 & -\omega\boldsymbol{\mu}_{tz}\end{bmatrix}, \quad (10)$$

Δ_ϵ and Δ_μ are the determinants of (the matrix representations of) the $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\mu}_t$ dyadics, respectively, and $\mathrm{Tr}\,(\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t)$ is the trace of (the matrix representation for) the dyadic $\mathbf{z}_0\times\boldsymbol{\mu}_t\cdot\mathbf{z}_0\times\boldsymbol{\varepsilon}_t$. Further, it can be shown that the longitudinal field components satisfy the following pair of (coupled) second-order differential equations (scalar eigenvalue problem):

$$\begin{bmatrix}\epsilon_zE_z\\ i\mu_zH_z\end{bmatrix}=\widehat{\mathfrak{B}}\frac{\mathfrak{A}}{D(\kappa)}\mathfrak{B}\begin{bmatrix}E_z\\ iH_z\end{bmatrix} \quad (11)$$

where $D(\kappa)$, $\mathfrak{A}$, $\mathfrak{B}$ are defined in (7)–(9) and:

$$\widehat{\mathfrak{B}}=\begin{bmatrix}-\omega\boldsymbol{\varepsilon}_{zt} & \mathbf{z}_0\times\nabla_t\\ \mathbf{z}_0\times\nabla_t & -\omega\boldsymbol{\mu}_{zt}\end{bmatrix}. \quad (12)$$

Note that, in general, $1/D(\kappa)$ does not commute with either $\mathfrak{B}$ or $\widehat{\mathfrak{B}}$ since these contain differentiation operations. The reader may verify that the result in (11) reduces to the equation given by Kales[2] for the special case of an axially magnetized gyromagnetic medium (*i.e.*, where ϵ is a scalar and $\boldsymbol{\mu}_{tz}=\boldsymbol{\mu}_{zt}=0$).

Any solution E_x, H_z to (11) yields, via (7), an eigenfunction (mode) of the transverse vector eigenvalue problem (5). This procedure is manifestly not valid when $D(\kappa)=0$. Therefore, the set of vector eigenfunctions obtained from all the solutions to (11) becomes complete only when we add such vector eigenfunctions of (5) which are admitted when $D(\kappa)=0$. That these additional eigenfunctions are the analogs of the TEM modes in the anisotropic case is evident from the fact that $D(\kappa)=(\omega^2\mu\epsilon-\kappa^2)^2$ for an isotropic medium with scalar μ and ϵ. The analogy to TEM modes indicated here should not be taken to imply any TEM-like properties of these eigenfunctions in the anisotropic case.

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* Received by the PGMTT, October 31, 1958. This note is based on a study undertaken pursuant to Contract AF-19(604)-2301 with the AF Cambridge Res. Center.

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An Extension of the Reflection Coefficient Chart to Include Active Networks*

INTRODUCTION

At a single frequency, a two-port can be represented by the scattering matrix [1], [5]

$$[b]=[S][a] \quad (1a)$$

$$b_1=s_{11}a_1+s_{12}a_2 \quad (1b)$$

$$b_1=s_{21}a_1+s_{22}a_2 \quad (1c)$$

where $s_{12}=s_{21}$ in the reciprocal two-port. If one defines an input reflection coefficient $\Gamma_{in}=b_1/a_1$ and a load reflection coefficient $\Gamma_L=a_2/b_2$ one can form

$$\Gamma_{in}=\frac{(s_{12}^2-s_{11}s_{22})\Gamma_L+s_{11}}{1-s_{22}\Gamma_L}. \quad (2)$$

Eq. (2) can be considered as a mapping of the Γ_L plane into the Γ_{in} plane. Since this is a bilinear transformation, angles between

* Received by the PGMTT, November 17, 1958.

intersecting lines are preserved and circles will transform into other circles. The load reflection coefficient can be written as $|\Gamma_L|e^{j\arg\Gamma_L}$, where for a passive load $|\Gamma_L|\leq 1$. At some output reference plane, the insertion of varying amounts of matched lossless line will vary the angle of Γ_L so that the locus of Γ_L will be a circle centered at the origin and with a radius of $|\Gamma_L|$. The locus of the input reflection coefficient will also be a circle which will not in general be centered at the origin of the Γ_{in} plane. If $|\Gamma_L|=1$, the Γ_{in} circle can be used to measure [2], [3] the scattering coefficients of the two-port. This circle may be referred to as the loss circle of the two-port. It is noted that if the load is dissipative (*i.e.*, $|\Gamma_L|<1$), the Γ_L circles will be concentric with the unity circle and will have smaller radii. The Γ_{in} circles will lie within the transformed unity circle, although the centers of the circles will not coincide but lie [4] on a straight line connecting the iconocenter and the origin of the Γ_{in} plane. If the angle of Γ_L is held constant (modulo π), this will describe a diameter of the unity circle in the Γ_L plane. This will map into the Γ_{in} plane as arcs of circles orthogonal to transformed constant $|\Gamma_L|$ circles. Since all of the diameters in the Γ_L plane intersected at the origin, all of the arcs in the Γ_{in} plane will intersect at the iconocenter.

Active Networks as Loads

If the restriction that the $|\Gamma_L|\leq 1$ is removed, corresponding to a source of power or negative resistance at the output, the extension of the theory follows logically. If $|\Gamma_L|$ is a constant and greater than unity, the locus will be a circle with a radius greater than unity (*i.e.*, outside the Smith Chart). It will be mapped into a circle which may or may not be all or in part outside the Smith Chart. If no part of the circle is outside the unity circle, an observer at the input port could not tell that there was an active element at the output. If part (or all) of the circle is outside of the chart the observer might (or would) see power coming out of the two-port's input, depending on the phase angle of Γ_L. If the angle of Γ_L is held constant (modulo π), the extension of the diameter in the Γ_L plane will map into the complete orthogonal circle as the magnitude of Γ_L varies from zero to infinity. It should be noted that the radii of the orthogonal circles will vary and one will have an infinite radius (a straight line). For every phase angle of Γ_L there will be a maximum $|\Gamma_{in}|$. Examining (2) it can be seen that if $s_{22}\Gamma_L=1$ the input reflection coefficient will become infinite. Since b_1/a_1 is infinite and a_1 is presumed to be finite, b_1 is infinite or infinite power is coming out of the input of the two-port. The value of Γ_L equal to the reciprocal of s_{22} for infinite power can also be obtained from (1b) as the power at the output becomes infinite. It is noted that letting Γ_L approach infinity will not represent infinite power except when $s_{22}=0$.

Active Networks in the Two-Port

Returning to the passive load, it might appear that if $|s_{22}|$ was equal to unity there was the possibility of obtaining infinite power from a passive two-port. From the conservation of energy in the passive two-port $|s_{12}^2|+|s_{22}^2|\leq 1$. Therefore $s_{12}=0$. Similarly, it can be shown by taking the equality[1] again $|s_{11}|=1$. When these values are substituted in (2), $\Gamma_{in}=s_{11}$. The two-port has been broken into two disjoint one-ports (no transmission between the two).

If the two-port contained some active elements the scattering coefficients could have any value. It is probable that the active elements will alter the reciprocity relationship $s_{12}=s_{21}$. However an equivalent reciprocal[2] s_{12}' could be determined by the Deschamps method $s_{12}'=\sqrt{s_{12}s_{21}}$. Therefore it is clear that active elements can be handled whether they appear in the load or the two-port.

Representation of Microwave Circuits

The bilinear transformation may be written as $\Gamma_{in}=T(\Gamma_L)$ where Γ_{in} and Γ_L represent the input and the output of the two-port, while the transformation T describes the two-port uniquely. Any Γ_{in} circle can be obtained from an infinite combination of T's and Γ_L's. The transformations T can be considered as belonging to three distinct types of transformations depending on whether $|\Gamma_L|$ is greater than, equal to, or less than unity. For a given loss circle a set of scattering coefficients can be determined. Only if $|\Gamma_L|=1$, the determined scattering coefficients will be the actual coefficients of the network. However, an observer at the input is unable to distinguish how the given loss circle is obtained and he can represent the two-port and the load as an "equivalent" two-port with a purely reactive load. If the load is purely reactive the "equivalent" network becomes the actual network.

If the other two parameters, Γ_L and T, are held constant (separately), added information may be found out about the behavior of microwave circuits. If the transformation T is held constant, the two-port is invariant, and the previous discussion regarding the transformation of Γ_L circles to Γ_{in} circles is applicable. The converse of the previous statement is also true since inverse transformation is also bilinear. If the load reflection circle is held constant, varying Γ_{in} will determine the transformation or the network.

Therefore it can be seen by the extension of the reflection coefficient chart that it is possible to represent any two-port and load at the input by another two-port with a purely reactive termination. Therefore a prescription of the Γ_{in} and Γ_L circles will determine a network. This description reduces to that of Deschamps when the Γ_L circle is the unit circle. The graphical method gives a clear geometric picture of the behavior of a given two-port in terms of input and output reflection coefficient loci.

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[1] If $|s_{12}|^2+|s_{22}|^2<1$, it would be impossible for $|s_{22}|$ to be unity and therefore $|s_{22}\Gamma_L|<1$.
[2] Only as far as an observer at the input is concerned.

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Characteristics of a Ferrite-Loaded Rectangular Waveguide Twist*

The Faraday effect in a straight rectangular waveguide, a section of which is completely filled with a ferrite material subjected to an axial magnetic field, has been described by Du Pré,[1] who states that, owing to the presence of a medium of dielectric constant and permeability greater than those of air, modes other than the usual TE_{10} mode may be propagated in the ferrite-filled section. In particular, the TE_{01} mode whose electric vector is perpendicular to the narrow dimension of the guide may be supported. If, owing to Faraday rotation, the TE_{10} mode is converted to the TE_{01} mode, propagation cannot take place beyond the ferrite-filled section. Experimentally this was confirmed by Du Pré who observed a minimum of transmitted power for 90° rotation. Similar results were obtained in this laboratory with a straight rectangular guide loaded with a cylindrical ferrite specimen the ends of which were tapered for matching purposes as shown in Fig. 1(a). For a given axial magnetic field the reduction of transmitted power was largest with the specimen in the center of the guide, but, as might have been expected, no nonreciprocal effects were observed. A twisted rectangular waveguide section, however, loaded with the same specimen, exhibited nonreciprocal characteristics. In the experiment the sample was mounted centrally midway between the flanges of a 90° commercial 0.4×0.9 inch twist and an axial magnetic field was applied as shown in Fig. 1(b).

For constant incident power the transmitted power varied with both the magnitude and the direction of the magnetic field. With the particular nickel-cobalt ferrite used, nonreciprocal behavior was most pronounced at around 8900 mc where, at the optimum value of field current, reversal of the magnetic field caused a reduction of

* Received by the PGMTT, November 24, 1958. This work was carried out under AEC Contract AT(30-1)-1909.
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